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Exact results for the critical behaviour of a Nienhuis $O(n)$ model on the square lattice

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Abstract. Exact results are obtained for a square lattice fifteen-vertex model and related $O(n)$ model recently proposed by Nienhuis. The partition function per site and finite-size corrections are obtained, along with the critical exponents via the Temperley–Lieb equivalent six-vertex model. The exact exponents $\nu = \frac{4}{7}$ and $\gamma = \frac{6}{7}$ follow in the limit $n = 0$ for even system sizes. Odd system sizes are also discussed. The fifteen-vertex model is also the vertex formulation of the anisotropic, or q-deformed, spin-1 biquadratic model.

1. Introduction

The study of vertex models has continued to flourish since Lieb's pioneering calculation of the residual entropy of square ice [1]. Indeed, the exact solution of the six-vertex model [2, 3] stands as one of the cornerstones in the theory of exactly solvable lattice models. For the six-vertex model in the critical regime, the vertex weights can be written (see, e.g., [3])

$$\omega_1, \ldots, \omega_6 = \rho(1, 1, x, x, 1 + x\mathrm{e}^{\mathrm{i}\lambda}, 1 + x\mathrm{e}^{-\mathrm{i}\lambda}) \tag{1}$$

where $x = \sin u / \sin(\lambda - u)$ and $\rho = \sin(\lambda - u)/\sin\lambda$.

The diagonal-to-diagonal transfer matrix is made up of operators

$$X_j = \rho\left(1 + xU_j\right) \tag{2}$$

each adding a vertex to the lattice. The operators U_j satisfy the celebrated Temperley–Lieb (TL) algebra [4, 5]:

$$\begin{aligned} &U_j^2 = \sqrt{Q}U_j \\ &U_jU_{j\pm1}U_j = U_j \\ &U_iU_j = U_jU_i \qquad |i - j| > 1 \end{aligned} \tag{3}$$

where, for the six-vertex model,

$$\sqrt{Q} = 2\cos\lambda. \tag{4}$$

Two other solvable vertex models are the three-state nineteen-vertex models of Zamolodchikov and Fateev [6] and Izergin and Korepin [7]. Now Nienhuis has recently proposed three solvable $O(n)$ models on the square lattice [8]. The second of these models is related to the Izergin–Korepin model and the third to the Zamolodchikov–Fateev model with fixed fugacity $n = 2$. In a special limit the Izergin–Korepin O(n) model reduces to a seven-vertex model on the honeycomb lattice. This vertex model is equivalent to the O(n) model originally proposed by Nienhuis [9] and has recently been investigated by a number of authors [10–12]. The critical properties of the more general Izergin–Korepin model have also been investigated [13–16].

One of the underlying motives for this work is that the $n = 0$ limit of the O(n) model yields information on the configurational statistics of two-dimensional polymers (see, e.g. [17, 18]). In this way the exact Nienhuis self-avoiding walk exponents $\nu = \frac{3}{4}$ and $\gamma = \frac{43}{32}$ [9] were recovered on both the honeycomb [11, 12] and square lattices [13, 15]. Moreover the square-lattice O(n) model contains a region akin to chain attraction in polymer solutions. In this region the exponents $\nu = \frac{12}{23}$ and $\gamma = \frac{53}{46}$ emerge as possible candidates for the critical exponents governing the Θ transition of two-dimensional polymers [15]. A different set of exponents for the Θ transition are due to Duplantier and Saleur [19] who obtained $\nu = \frac{4}{7}$ and $\gamma = \frac{8}{7}$ via Coulomb gas arguments for an O(n) model on the honeycomb lattice with random defects.

Exact results are obtained here for the critical behaviour of the first of the three square-lattice O(n) models introduced by Nienhuis [8]. Our approach is to adopt the philosophy used in the study of the other O(n) models [11–13, 15] and investigate the related vertex model. The critical exponents are obtained via the dominant finite-size behaviour. The notion that conformal invariance reflects the operator content of a model in its finite-size behaviour has breathed new life into the study of solvable lattice models (see, e.g. [20] and the reprint volumes [21, 22]). A numerical investigation into the finite-size behaviour of the loop version of the first Nienhuis square lattice O(n) model is given in Blöte and Nienhuis [23], where it is labelled as branch 0.

2. The various models

We begin by recalling the definition of the models [8, 23]. The configurations of the loop model are the graphs G consisting of non-intersecting closed polygons covering some (or none) of the edges of the square lattice. Each vertex is thus visited either none, one or two times. The nine allowed vertices for the general loop model are shown in figure 1. For the model of interest here, the corresponding weights are

$$\rho_1, \ldots, \rho_9 = (v + w, v, v, w, w, 0, 0, v, w) \tag{5}$$

where Blöte and Nienhuis have investigated the symmetric point $v = w = 1$ [23]†. The partition sum is defined by

$$Z = \sum_G \rho_1^{m_1} \cdots \rho_9^{m_9} n^N \tag{6}$$

where m_i is the number of vertices of type i and N is the number of loops of fugacity $n = 2\cos\lambda - 1 = e^{2i\theta} + e^{-2i\theta}$ in a given configuration G (thus $m_6 = m_7 = 0$). In more recent work, on multi-coloured loop models, this model is referred to as a *dilute* loop model, in contrast to the *dense* loop model in which ρ_8 and ρ_9 are the only non-zero weights [24].

† Actually they investigate the point $v = w = \frac{1}{2}$, so we have introduced a normalization factor of 2 per vertex.

Figure 1. The vertex states of the loop model.

The present loop model can also be viewed as a two-colour model constructed from the dense loop model [25].

On the other hand, Nienhuis [8] has cast the above partition sum into that of a vertex model,

$$Z = \sum \prod (\text{vertex weights}) \tag{7}$$

where the summation is now over all allowed arrow coverings of the lattice. All nineteen possible vertices are depicted in figure 2. The vertex weights corresponding to (5) can be written [8]†

$$\begin{aligned}
&\omega_1 = v + w \\
&\omega_2 = \omega_3 = \omega_4 = \omega_5 = v \\
&\omega_6 = \omega_8 = e^{i\theta} w \\
&\omega_7 = \omega_9 = e^{-i\theta} w \\
&\omega_{10} = \omega_{11} = \omega_{12} = \omega_{13} = 0 \\
&\omega_{14} = \omega_{15} = v \\
&\omega_{16} = \omega_{17} = w \\
&\omega_{18} = v + e^{-2i\theta} w \\
&\omega_{19} = v + e^{2i\theta} w .
\end{aligned} \tag{8}$$

In order to make contact with the numerical results of [23] we consider for simplicity only the isotropic case $v = w = 1$. Although important for deriving the related quantum spin chain, this anisotropy does not change the underlying physics.

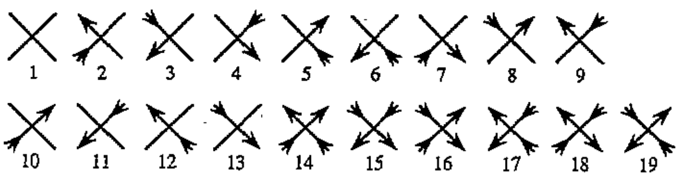

Figure 2. The states of the nineteen-vertex model.

† Note that we have made an explicit choice $\chi = -\theta/2$ for the gauge factor χ appearing in [8].

Given the vertex weights (8) it follows that the equivalent result to (2) for the isotropic fifteen-vertex model is

$$X_j = 1 + U_j \tag{9}$$

where

$$U_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad V = \begin{pmatrix} q^2 & q & 1 \\ q & 1 & q^{-1} \\ 1 & q^{-1} & q^{-2} \end{pmatrix} \tag{10}$$

and $q = e^{i\theta}$ satisfies the TL algebra (3) with

$$\sqrt{Q} = 1 + n. \tag{11}$$

This directly establishes the TL equivalence between the fifteen- and six-vertex models. This equivalence has been established in [23] by other arguments. In particular, it implies that the corresponding O(n) model is critical when $n \leqslant 1$ with a gap opening up in the eigenspectrum for $n > 1$. We shall confine our attention to the critical region.

We have thus seen that the vertex model with weights given in (8) provides a three-state representation of the TL algebra. The anisotropic parametrization of this three-state model can be found in the paper by Deguchi *et al* [26] on the relationship between solvable lattice models and the theory of knots and braids (see, in particular, equation (A.4)). The related quantum spin chain is the q-deformed generalization [27] of the isotropic spin-1 biquadratic model [28]. The Hamiltonian for this family of quantum spin chains is simply the sum of the TL operators U_j. A numerical investigation of the q-deformed spin chain has recently been undertaken by Alcaraz and Malvezzi [29] (see also [30]). We note that a Bethe ansatz solution of the fifteen-vertex TL model, along with the q-deformed spin chain, has been given by Klümper via a mapping onto the six-vertex model [31]. However, we shall not need such a solution here. The TL equivalence between the various models will suffice. In particular, we make extensive use of the known finite-size properties of the six-vertex model/XXZ chain (see, e.g. [12, 32–34] and references therein) to obtain the results of interest for the fifteen-vertex model and thus finally the O(n) model.

3. Critical behaviour

From the TL equivalence between the fifteen- and six-vertex models, the bulk free energy per site of the loop model immediately follows from the isotropic six-vertex model where $u = \frac{1}{2}\lambda$ (see, e.g., [3, 5])†,

$$f_\infty = \int_\infty^\infty \frac{\sinh(\pi - \lambda)t \sinh \lambda t}{2t \sinh \pi t \cosh \lambda t} \mathrm{d}t \tag{12}$$

where from (4) and (11) $n = 2\cos\lambda - 1$ is the most convenient parametrization for the O(n) model (rather than $n = 2\cos 2\theta$). Some special values are [4]:

(1) $f_\infty = 2\ln[\Gamma(\frac{1}{4})/2\Gamma(\frac{3}{4})] = 0.783\ldots$ at $n = 1$ ($\lambda = 0$);

† Note that to make all three models share the same bulk free energy we drop the normalization factor ρ appearing in the vertex weights (1).

(2) $f_\infty = \ln 2 = 0.693\ldots$ at $n = 0$ ($\lambda = \frac{1}{3}\pi$);
(3) $f_\infty = e^{2C/\pi} = 0.583\ldots$ at $n = -1$ ($\lambda = \frac{1}{2}\pi$), where C is Catalan's constant;
(4) $f_\infty = \frac{3}{2}\ln(\frac{4}{3}) = 0.431\ldots$ at $n = -2$ ($\lambda = \frac{2}{3}\pi$), where e^{f_∞} is Lieb's residual entropy of square ice [1].

To obtain the critical exponents of the loop model, our starting point is a direct numerical comparison of the eigenspectra of three transfer matrices:

(1) the diagonal-to-diagonal transfer matrix of the loop model (5) with periodic boundary conditions and dangling bonds used in [23];
(2) the row-to-row transfer matrix of the fifteen-vertex model (8);
(3) the row-to-row transfer matrix of the six-vertex model (1).

Each transfer matrix is defined on a finite strip of width L. In the latter two cases we also impose a seam $e^{i\phi}$. Specifically, the seam is a line parallel with the transfer matrix direction crossing a column of horizontal bonds. An extra weight $e^{i\phi}$ ($e^{-i\phi}$) is assigned for right (left) pointing arrows crossing the seam.

The loop model transfer matrix has been discussed at length in [23]. Two sectors of this matrix are labelled by $n_d = 0$ and $n_d = 1$ which are, respectively, even- and odd-parity sectors. The $n_d = 0$ sector is of size $a_L \times a_L$ and the $n_d = 1$ sector is of size $b_L \times b_L$, where the connectivities a_L and b_L ($a_L < b_L$) are tabulated in table 6 of [23]. In contrast, the fifteen-vertex transfer matrix is $3^L \times 3^L$ while the six-vertex transfer matrix is $2^L \times 2^L$. Each of these transfer matrices decomposes into sectors determined in size by the trinomial and binomial distributions.

Our observations on the eigenspectra of the three models for even L are as follows. For given n, all eigenvalues in the $n_d = 0$ sector of the loop model occur in the largest sector of the fifteen-vertex model at the corresponding value of θ with seam value $\phi_{15} = 2\theta$ (recall that $n = 2\cos 2\theta$). All eigenvalues in the $n_d = 1$ sector of the loop model occur in the second largest sector of the fifteen-vertex model with $\phi_{15} = 0$, i.e. with periodic boundary conditions. In contrast to the $n_d = 0$ case, the $n_d = 1$ sector is the same size as the next-largest sector of the fifteen-vertex model and the mapping between eigenvalues in this case is one to one. The relationship between the eigenspectra of the two vertex models is more complicated. All eigenvalues in the largest sector of the six-vertex model appear as eigenvalues in the largest sector of the fifteen-vertex model when the seams obey

$$\cos\phi_6 = \cos\phi_{15} + \tfrac{1}{2}. \tag{13}$$

This relationship was observed between the eigenspectra of the q-deformed spin-1 biquadratic and spin-$\frac{1}{2}$ XXZ chains with twisted boundary conditions [29]. Conversely, the relationship between eigenvalues in the next-largest sector of each model holds with no seam, i.e. $\phi_6 = \phi_{15} = 0$.

We are now in a position to obtain the central charge and scaling dimensions of the various models. Defining the free energy per site as $f_L = L^{-1}\ln\Lambda_0$, where Λ_0 is the largest eigenvalue of the transfer matrix, the central charge follows from the leading finite-size correction [35, 36]

$$f_L \simeq f_\infty + \frac{\pi c}{6L^2}. \tag{14}$$

The central charge of the periodic six-vertex model is $c = 1$ with, more generally [32, 34],

$$c = 1 - \frac{6\phi_6^2}{\pi(\pi - \lambda)}. \tag{15}$$

From (13) with $\phi_{15} = 0$ the corresponding seam in the six-vertex model is $e^{i\phi_6} = \frac{1}{2}(3+\sqrt{5})$ and thus from (15) the central charge of the periodic fifteen-vertex model is

$$c = 1 + \frac{6[\ln\frac{1}{2}(3+\sqrt{5})]^2}{\pi(\pi - \lambda)}. \tag{16}$$

This result is in agreement with that obtained for the spin-1 q-deformed biquadratic model in the critical region [29] (there is in fact a misprint in equation (2.20) of [29]). We thus see that, in contrast to the six-vertex model, the central charge of the periodic fifteen-vertex model is continuously varying with λ. The exact central charge of the O(n) model follows from (13) with $\phi_{15} = 2\theta$, which gives $\cos\phi_6 = \cos\lambda$ and thus from (15)

$$c_{O(n)} = 1 - \frac{6\lambda^2}{\pi(\pi - \lambda)}. \tag{17}$$

This result also follows immediately from the two-colour interpretation [25].

The scaling dimensions X_i are related to the inverse correlation lengths via (see, e.g., [36] and references therein)

$$\xi_i^{-1} = \ln(\Lambda_0/\Lambda_i) \simeq 2\pi X_i/L \tag{18}$$

where ξ_i is the correlation length of the ith operator and Λ_i is the ith leading eigenvalue. The λ-dependent scaling dimensions appearing in the six-vertex model are of the form [32–34]

$$x_{n,m} = n^2 x_p + \frac{m^2}{4x_p} \quad \text{where} \quad x_p = \frac{\pi - \lambda}{2\pi}. \tag{19}$$

Along with the central charge Blöte and Nienhuis have given numerical estimates of the thermal and magnetic scaling dimensions. The thermal scaling dimension X_ϵ of the loop model follows from the leading excitation in the $n_d = 0$ sector [23]. This state is the leading excitation in the largest sector of the six-vertex model with seam $\phi_6 = \lambda$. Hence [32]

$$X_\epsilon = x_{0,1-\lambda/\pi} - x_{0,\lambda/\pi} = \frac{\pi - 2\lambda}{2(\pi - \lambda)}. \tag{20}$$

The magnetic scaling dimension X_σ follows from the leading eigenvalue in the $n_d = 1$ sector [23]. This eigenvalue is equivalent to the leading eigenvalue in the next-largest sector of the six-vertex model with $\phi_6 = 0$. Hence [32]

$$X_\sigma = x_{1,0} - x_{0,\lambda/\pi} = \frac{\pi - 2\lambda}{2(\pi - \lambda)}. \tag{21}$$

These exact results are in agreement with the identification with the Kac formula: $X_\epsilon = X_\sigma = 2\Delta(m, m)$ with $m + 1 = \pi/\lambda$ [23] where (see, e.g., [20])

$$c = 1 - \frac{6}{m(m+1)} \qquad \Delta(p, q) = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}. \tag{22}$$

That $X_\epsilon = X_\sigma$ for this model is due to the equality of the relevant eigenvalues for finite L. This equality is a consequence of the larger degeneracy between the eigenvalues of the two sectors $n_d = 0$ and $n_d = 1$. We find that the next-to-leading thermal and magnetic exponents are given by

$$X_{\epsilon\epsilon} = X_{\sigma\sigma} = x_{2,0} - x_{0,\lambda/\pi} = \frac{4(\pi-\lambda)^2 - \lambda^2}{2\pi(\pi-\lambda)} \tag{23}$$

which confirms the identification $X_{\epsilon\epsilon} = 2\Delta(m, m-1)$ made in [23]. More generally, we find that the above scaling dimensions belong to the sequence

$$X_j = x_{j,0} - x_{0,\lambda/\pi} = 2\Delta(0, j) \tag{24}$$

with $X_\epsilon = X_\sigma = X_1$ and $X_{\epsilon\epsilon} = X_{\sigma\sigma} = X_2$.

When L is odd the equivalence between the largest eigenvalues of the three models is independent of the seam. Thus in this case the central charge is that of the six-vertex model [32], namely

$$\tilde{c} = -\tfrac{1}{2} + 3\lambda/2\pi. \tag{25}$$

Consequently the scaling dimension associated with the 'interface energy' between odd and even sites [23] is given by

$$X_{\text{int}} = [c_{O(n)} - \tilde{c}]/12 = \frac{(\pi-\lambda)^2 - 4\lambda^2}{8\pi(\pi-\lambda)}. \tag{26}$$

Here we make the identification $X_{\text{int}} = 2\Delta(\frac{1}{2}m, \frac{1}{2}m) = 2\Delta(0, 1/2)$ which differs from that made in [23]. In the six-vertex model the largest eigenvalue is two-fold degenerate for L odd. In this case an excitation of spin-wave index $\frac{1}{2}$ produces a 'defect' with the associated scaling dimension $x_{1/2,0}$ perturbing the $c = 1$ theory to one with effective central charge given by equation (25). We see that the largest eigenvalue of the fifteen-vertex model is three-fold degenerate for L odd. On the other hand, there is a two-fold degeneracy in the loop model, with the largest eigenvalue appearing in both the $n_d = 0$ and $n_d = 1$ sectors. This degeneracy, and subsequent vanishing of the associated scaling amplitude, can be established in both sectors by general arguments for odd system sizes and $n = 0$ [37]. Similar arguments predict the vanishing of the central charge at $n = 0$ for L even [37]. For L odd we find the leading excitation has scaling dimension $X = 1 - \lambda/\pi$.

The critical exponents of the loop model follow in the usual way, with $\eta = 2X_\sigma$, $1/\nu = 2 - X_\epsilon$ and $\gamma = (2-\eta)\nu$. Thus for L even we have the exponents $\nu = \frac{4}{7}$ and $\gamma = \frac{6}{7}$ in the limit $n = 0$. These are to be compared with the Duplantier–Saleur values ($\nu = \frac{4}{7}$ and $\gamma = \frac{8}{7}$) [19]. The above exponents are also distinct from other known universality classes, including kinetic growth trails ($\nu = \frac{4}{7}$ and $\gamma = 1$) and trails on the L lattice, which are equivalent to SAWs on the Manhattan lattice and thus in the SAW universality class (see, e.g., [19, 38] and references therein).

The precise elucidation of the present universality class, and in particular whether or not it describes the configurational statistics of self-avoiding self-attracting trails with a turn at every step, remains an interesting question for further study.

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